Noether's First Theorem
Part Two

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Recap From the Last Talk

There are three types of quantities we will use today to define the state of a physical system:

- time, a real number denoted by t
- geometrical variables generally denoted by individual coordinates $q^1, \ldots, q^N$, or simply the configuration vector $q \in \mathbb{R}^N$.

  (In more general situations than those considered today q could correspond to a point in a coordinate patch of an N-dimensional manifold, the configuration manifold.)
- dynamical variables generally denoted by $\dot{q}^1, \ldots, \dot{q}^N$, or simply $\dot{q}$, also in $\mathbb{R}^N$.

  (Such a $\dot{q}$ could represent a vector in the tangent space at a point q of a configuration manifold.)

If a function has name q the symbols $q(t)$ indicate one of its values, not the whole function. However it is standard to use the locution "the function $q(t)$" to emphasize the symbol we intend to use for the independent variable, and I don't intend to fight this practice. You simply must deduce whether $q(t)$ is to be a point or the whole function from context.

A prediction of the evolution of a physical system will correspond to a time-dependent path $q(t)$ through various "configuration vector" values.

Associated with this path will be a tangent vectors $\frac{dq}{dt}(t)$ which will correspond to a "dynamical vectors" $\dot{q}(t)$.

Using such a path, at each time we can produce a point

$$(t, q(t), \dot{q}(t)) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N.$$

Consistent

Specific coordinate systems are a convenience, not part of nature. Many coordinate systems can be used and an "answer," a number or vector such as momentum produced to describe something real using these coordinates, should be consistent.

That is to say: physical law is to be associated with a collection of preferred coordinate systems, and must be organized in such a way that when predictions are made using the law in two of the preferred coordinate systems and *after interpretation to account for different coordinates*, both calculations produce an equivalent "answer."

Maxwell's Theory of Electromagnetism with Galilean coordinate changes—Galilean Relativity—notoriously fails this test.

Invariant

The phrase *invariant under an "admissible" coordinate change* here means that the number or vector, though calculated in coordinates, need not be "interpreted." It is actually left unaltered when you change to this new "admissible" coordinate system.

The "admissible" coordinate changes we have in mind will be translations, or rotations, or a change in the origin of time, though our results will apply in other cases too. These collections of coordinate changes will, each, form groups that can be described by continuous parameters and represent, geometrically, symmetries in the space on which they act.

(Actually, the full Lie group structure is not used in our work. Only coordinate changes corresponding to very small or "infinitesimal" values of the parameter were used by Noether and needed by us.)

Conserved

The word *conserved* on the other hand refers to equal numerical measurements that are obtained within a single reference frame over a time interval, such as the total momentum of particles before and after some interaction.

They don't imply that this number will be the same in different coordinates.

Noether's Theorems tell us when and how invariance of a functional called the action under a continuous symmetry group of coordinate changes is related to conservation laws on physically realized paths in the configuration space.

A Functional

All functions will be presumed to have whatever degree of continuity/differentiability required for a given calculation. For instance Taylor series expansions of functions are allowed if that should become convenient.

Consider the function

$$L\colon \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}.$$

In the simplest applications L might just be $T - V$, the difference between kinetic and potential energy of a system of particles.

In that example L would have units of energy, $[\mathrm{kg\,m^2\,s^{-2}}]$

(In a more general setting L will be defined on $\mathbb{R} \times TM$ where TM is the tangent manifold of the configuration manifold M. This is the setting of General Relativity with its spacetime manifold, and also a natural way to handle problems with constraints.)

We will let $\mathcal{S}$ be a specified set of functions of the form

$$q\colon (a,b) \to \mathbb{R}^N$$

where (a,b) is a generic (i.e. not fixed for all members of $\mathcal{S}$) interval in $\mathbb{R}$.

Members of $\mathcal{S}$ are potential configuration paths[1] and for the simplest examples the coordinates of q might have units of [m].

The derivatives of the functions in $\mathcal{S}$, of the form

$$\dot{q}\colon (a,b) \to \mathbb{R}^N,$$

would then have units $[\mathrm{m\,s^{-1}}]$.

$\mathcal{S}$ must be a pretty rich class of functions, containing many vector spaces of functions.[2]

[1] These configuration paths are not restricted to interpretation as rectangular coordinate paths. Kinetic energy therefore *cannot be presumed* to be given in terms of $\dot{q} \cdot \dot{q}$!

[2] For instance, the twice continuously differentiable functions on various domain time intervals.

We define $\Gamma\colon \mathcal{S} \to \mathbb{R}$ to be

$$\Gamma(q) = \int_a^b L(t, q(t), \dot{q}(t))\, dt$$

where $\dot{q}(t) = \frac{dq}{dt}(t)$.

The time dependence of q and $\dot{q}$, and even a reminder that there is an explicit path from $q(a)$ to $q(b)$, may be suppressed as in

$$\Gamma = \int_a^b L(t, q, \dot{q})\, dt = \int_a^b L\, dt.$$

Γ is called a *functional* (never linear!) and $\Gamma(q)$ is sometimes called the *action integral* of the path q on the interval $[a,b]$.

The action integral often has units $[\mathrm{kg\,m^2\,s^{-1}}]$.

Notation Recap

We have used variables t, q and $\dot{q}$ and a function $L(t,q,\dot{q})$ and functions $q(t)$ and $\frac{dq}{dt}(t) = \dot{q}(t)$.

If you see in this or similar situations a notation such as $\frac{\partial}{\partial t}$ or $\frac{\partial}{\partial \dot{q}^i}$ it means the function to which it applies is assumed *for the purposes of the initial calculation* to have t or $\dot{q}^i$ as one of its variables and other variables are constant. After that partial derivative is found, a path $q(t)$ and its derivative $\dot{q}(t)$ might be substituted to create a function of t.

$$\frac{\partial L}{\partial t} = \frac{\partial}{\partial t} L(t,q,\dot{q}) \quad \text{and} \quad \frac{\partial}{\partial \dot{q}^i} L(t,q,\dot{q})$$

are presumed for purposes of the calculation to have t dependency *only* in its first coordinate and $\dot{q}^i$ dependency in the $i + N + 1$ coordinate only, even if you have a path $q(t)$ in mind!

Notation Recap

However a total derivative such as $\frac{dL}{dt} = \dot{L}$ will indicate that a path $q(t)$ has been selected (even if it is not shown here!) and fed to L before differentiating that composite function:

$$\dot{L}(t) = \frac{d}{dt} L(t, q(t), \dot{q}(t)).$$

This function of t must be calculated by the chain rule:

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q^i}\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i$$

where we use the Einstein summation convention to indicate summation over all values of an index variable in terms containing a product of exactly two terms with the same index, one "high" and the other "low."

Suppose q is a path, a potential description of a physical evolution from configuration $q(a)$ at time a to configuration $q(b)$ at time b , and ζ is a path with $\zeta(a) = 0 = \zeta(b)$.

Define, for all real ϵ in some neighborhood of 0 the function Q_ϵ by $Q_\epsilon = q + \epsilon\,\zeta$.

We can define

$$\Gamma_\epsilon = \int_a^b L(t, Q_\epsilon, \dot{Q}_\epsilon)\, dt = \int_a^b L(t, q + \epsilon\,\zeta, \dot{q} + \epsilon\,\dot{\zeta})\, dt.$$

So $q = Q_0$, and of course then $\Gamma = \Gamma_0$.

We say q is a stationary path with respect to ζ provided

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Gamma_\epsilon = \lim_{\epsilon\to 0} \frac{\Gamma_\epsilon - \Gamma_0}{\epsilon} = 0,$$

and define q to be a stationary path if it is stationary with respect to every possible ζ.

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The Euler-Lagrange Equations

We define the *Euler-Lagrange operator* $\psi = (\psi_1, \ldots, \psi_N)$ on function space $\mathcal{S}$ by

$$\psi_i(q) = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \qquad i = 1, \ldots, N.$$

We found in Part One of these talks that under our smoothness conditions a path q is stationary when and only when $\psi(q)$ is the constant-zero function:

$$\psi_i(q) = \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0 \qquad i = 1, \ldots, N.$$

(Note the total derivative $\frac{d}{dt}$. These partial derivatives are to be taken at the implied coordinate and subsequently evaluated at $(t, q(t), \dot{q}(t))$ to produce functions of time.)

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$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \qquad i = 1, \ldots, N$$

constitutes a system of N second order ordinary differential equations in the N functions $q_1, \ldots, q_N$ of variable t.

Expanding $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}$ we have the explicit second order system

$$\frac{\partial L}{\partial q^i} = \frac{\partial^2 L}{\partial t \partial \dot{q}^i} + \frac{\partial^2 L}{\partial q^\mu \partial \dot{q}^i}\dot{q}^\mu + \frac{\partial^2 L}{\partial \dot{q}^\mu \partial \dot{q}^i}\ddot{q}^\mu \qquad i = 1, \ldots, N.$$

This is the Euler-Lagrange System (ELE) of ordinary differential equations. They can, with appropriate boundary conditions, be solved for q numerically or, in a few very-special but often-presented cases, exactly.

So in one sense, the business of analytical mechanics is done. We can predict the configuration path to any desired accuracy, or exactly in some cases.

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The equations

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \qquad i = 1, \ldots, N$$

suggest that the time-rate-of-change of $p_i = \frac{\partial L}{\partial \dot{q}^i}$ is important and we call the quantity p_i, which in our simplest example case has units $[\mathrm{kg\,m\,s^{-1}}]$, a generalized momentum.

p_i is said to be the canonical momentum *conjugate* to the generalized coordinate q^i.

With this definition, the ELE System in "p dot" form is

$$\frac{\partial L}{\partial q^i} = \dot{p}_i \qquad i = 1, \ldots, N.$$

We can see that the number p_i is preserved on a solution path (i.e. the constant function on the path) precisely when L is independent of the coordinate corresponding to q_i around the path q. q_i is called a *cyclic coordinate* when that happens.

We also found that

$$\frac{\partial L}{\partial t} = -\frac{d}{dt}\left(p_i \dot{q}^i - L\right).$$

So the time derivative of the function

$$H = p_i \dot{q}^i - L$$

seems to be important.

We can see that this number is preserved on a solution path precisely when L is independent of t around the path q.

H is called the Hamiltonian of the path in the physical scenario described by L.

We have the following "H dot" implication of the ELE system evaluated along a solution path:

$$\frac{\partial L}{\partial t} = -\dot{H}.$$

The ELE "H dot" and "p dot" equations

$$\frac{\partial L}{\partial t} = -\dot{H} \quad \text{and} \quad \frac{\partial L}{\partial q^i} = \dot{p}_i \quad i = 1, \ldots, N$$

tell us when the canonical momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$ and the Hamiltonian $H = p_i \dot{q}^i - L$ will be constants along solution paths.

This will happen when time is not explicitly present in the Lagrangian, or when L is independent of one of the geometrical coordinates: that is, when one of the coordinates is cyclic.

This independence corresponds to continuous translational symmetry in time or the configuration space under which L is invariant.

We have discovered a special case of the Noether Theorems.

Note:

We have discovered conditions on L and $q(t)$ for which the path $q(t)$ is stationary, which is itself a necessary condition for it to be a solution path for a physical system in many cases of interest.

In the next section we will deal with conditions of invariance of the action, and there *we will not* assume the path is stationary: we will work with a generic path and find conditions that insure the invariance of the action integral on the path when that path is shifted according to a family of transformations.

It is Noether's Theorem which puts these conditions together to create constants along solution paths.

A Continuous Group of Transformations

Emmy Noether was an expert on the technologies associated with Lie Groups, and to her the theorems we are exploring were facts about Continuous Lie Groups of Coordinate Transformations. We will not look at Lie Groups explicitly (a great topic for later talks) but will extract just those features about them needed for these theorems.

First, we will consider general invertible transformations

$$T_\varepsilon \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \times \mathbb{R}^N \qquad (t, q) \longrightarrow (t_\varepsilon, q_\varepsilon)$$

defined for real parameter ε, any one of all values in some (possibly tiny) interval around 0 and with the property that T_0 is the identity map on $\mathbb{R} \times \mathbb{R}^N$ and

$$\lim_{\varepsilon \to 0} T_\varepsilon(t, q) = (t, q) \quad \text{for all } (t, q) \in \mathbb{R} \times \mathbb{R}^N.$$

As we did with L and the members of $\mathcal{S}$, we presume as much differentiability of T_ε as we require for a calculation, *including* differentiability with respect to ε.

Any path q from $\mathcal{S}$ has a graph which is a collection of points $(t, q(t))$ in $\mathbb{R} \times \mathbb{R}^N$ and T_ε shifts each such point to a new point

$$T_\varepsilon(t, q(t)) = (t_\varepsilon(t, q(t)),\ q_\varepsilon(t, q(t))) \in \mathbb{R} \times \mathbb{R}^N.$$

These points form the graph of a new path defined on the interval $(a', b') = (t_\varepsilon(a, q(a)), t_\varepsilon(b, q(b)))$ and this new function

$$Q_\varepsilon \colon (a', b') \to \mathbb{R}^N$$

has values $Q_\varepsilon(w_\varepsilon) = q_\varepsilon(t, q(t))$ when $w_\varepsilon = t_\varepsilon(t, q(t))$.

In other words, $Q_\varepsilon \circ w_\varepsilon(t) = q_\varepsilon(t, q(t))$.

For each ε we define functional Φ_ε on our family of paths $\mathcal{S}$ by

$$\Phi_\varepsilon(q) = \int_{a'}^{b'} L\left(w_\varepsilon, Q_\varepsilon(w_\varepsilon), \frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon)\right) dw_\varepsilon$$

where $a' = t_\varepsilon(a, q(a))$ and $b' = t_\varepsilon(b, q(b))$.

Each specific path q creates a new path Q_ε with a different time parameter given by w_ε whose values vary from a' to b'.

The two time parameters

$$w_\varepsilon \longleftrightarrow t$$

are related to each other via invertible and differentiable functions. When regarded as a function of t, we have

$$w_\varepsilon(t) = t_\varepsilon(t, q(t)).$$

$\Phi_0(q) = \Gamma(q)$, the action integral on path q.

$\Phi_\varepsilon(q)$ will be close to $\Gamma(q)$ for small ε, given our smoothness assumptions on everything in sight, but *how* close?

It looks like

$$\Phi_\varepsilon(q) = \int_{a'}^{b'} L\left(w_\varepsilon, Q_\varepsilon(w_\varepsilon), \frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon)\right) dw_\varepsilon$$

might be simply related to

$$\Phi_0(q) = \int_a^b L\left(t, q(t), \dot{q}(t)\right) dt$$

by some kind of a "change-of-variable" formula, but *this is not the case.*

Consider a simple example where $w_\varepsilon(t) = t$ and $Q_\varepsilon(t) = q(t) + \varepsilon v$ for a fixed vector v.

$$\Phi_\varepsilon(q) = \int_a^b L\left(t, q(t) + \varepsilon v, \dot{q}(t)\right) dt.$$

The path $q(t) + \varepsilon v$ is shifted away from $q(t)$ and there is no reason, with our conditions on L, that the two integrands, though close for tiny ε, should be otherwise related.

The difference $\Phi_\varepsilon(q) - \Phi_0(q)$ can be calculated as

$$\begin{aligned}
&\Phi_\varepsilon(q) - \Phi_0(q) \\
&\quad = \int_{a'}^{b'} L\left(w_\varepsilon, Q_\varepsilon(w_\varepsilon), \frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon)\right) dw_\varepsilon - \int_a^b L\left(t, q(t), \dot{q}(t)\right) dt \\
&\quad = \int_a^b \left[L\left(w_\varepsilon(t), Q_\varepsilon(w_\varepsilon(t)), \frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon(t))\right) \frac{dw_\varepsilon}{dt} - L\left(t, q(t), \dot{q}(t)\right)\right] dt \\
&\quad = \int_a^b \left(L_\varepsilon \frac{dw_\varepsilon}{dt} - L\right) dt
\end{aligned}$$

where we define L_ε as the specific indicated function of time—that is, L composed with the shifted path as indicated.

We will need to look more carefully in a minute at L_ε, so let's begin that process now. L_ε was created from q and T_ε.

So $L_\varepsilon = L_\varepsilon(t)$ depends specifically on ε and q.

Noting $Q_\varepsilon(w_\varepsilon) = q_\varepsilon(t, q(t))$ when $w_\varepsilon = t_\varepsilon(t, q(t))$ we have

$$\begin{aligned} L_\varepsilon(t) &= L\left(w_\varepsilon(t), Q_\varepsilon(w_\varepsilon(t)), \frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon(t))\right) \\ &= L\left(t_\varepsilon(t, q(t)), q_\varepsilon(t, q(t)), \frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon(t))\right). \end{aligned}$$

Calculating $\frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon(t))$ will be tricky, and we will need to do that. When we get more structure in our transformations we will get at it by noting that

$$\frac{dQ_\varepsilon}{dw_\varepsilon} = \frac{d}{dw_\varepsilon} Q_\varepsilon \circ w_\varepsilon \circ t = \frac{dQ_\varepsilon \circ w_\varepsilon}{dt}\frac{dt}{dw_\varepsilon} = \frac{dQ_\varepsilon \circ w_\varepsilon}{dt} \Big/ \frac{dw_\varepsilon}{dt}. \quad (1)$$

Back to our main track, we have created the expression for the difference $\Phi_\varepsilon(q) - \Phi_0(q)$ as

$$\Phi_\varepsilon(q) - \Phi_0(q) = \int_a^b \left(L_\varepsilon \frac{dw_\varepsilon}{dt} - L\right) dt.$$

We say the action integral $\Gamma = \Phi_0$ is invariant under this family of transformations provided

$$\lim_{\varepsilon \to 0} \frac{\Phi_\varepsilon(q) - \Phi_0(q)}{\varepsilon} = 0.$$

or, equivalently, if for every q

$$\left.\frac{d\Phi_\varepsilon(q)}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

Because q—and hence the interval (a, b)—is arbitrary[3], the condition on

$$\int_a^b \left(L_\varepsilon \frac{dw_\varepsilon}{dt} - L\right) dt$$

is equivalent to the same restriction applied pointwise to each integrand (that is, for each t) and this condition is easier to apply so we adopt it.

Γ is therefore said to be invariant under this family of transformations provided the invariance condition is satisfied:

$$\lim_{\varepsilon \to 0} \frac{L_\varepsilon \frac{dw_\varepsilon}{dt} - L}{\varepsilon} = 0$$

for each q at each time in the domain of q.

[3]The unlimited differentiability condition on T_ε implies that $L_\varepsilon \frac{dw_\varepsilon}{dt}$ has continuous derivative with respect to ε and also allows us to apply Leibnitz's integral rule and differentiate past the integral sign.

We now get more specific about our family of transformations by positing that t_ε and q_ε take a certain form.

We assume

$$t_\varepsilon = t + \varepsilon\tau + \varepsilon^2 f \quad \text{and} \quad q_\varepsilon = q + \varepsilon\zeta + \varepsilon^2 g$$

where $\tau = \tau(t, q)$ and $f = f(t, q)$ are real functions and $\zeta = \zeta(t, q)$ and $g = g(t, q)$ are functions into $\mathbb{R}^N$, and all four are fixed functions—that is, independent of ε—and therefore characterize the whole family of transformations.

We will focus on τ and ζ, and say they generate this family of transformations.

$$t_\varepsilon = t + \varepsilon\tau + \varepsilon^2 f \quad \text{and} \quad q_\varepsilon = q + \varepsilon\zeta + \varepsilon^2 g.$$

The existence of generators for a continuous Lie group of coordinate transformations is a theorem of that subject, and that is all we need to know about Lie groups here.

In that case f and g are actually determined by τ and ζ, but that will not concern us.

In fact, Noether's result does not require consideration of a whole Lie group. Only coordinate transformations that move paths by "infinitesimal" amounts, corresponding to a "tiny" interval of ε around 0, are required.

f and g will be irrelevant on such a tiny ε-interval.

Suppose $t_\varepsilon = t + \varepsilon\tau + \varepsilon^2 f$ and $q_\varepsilon = q + \varepsilon\zeta + \varepsilon^2 g$ is a family of transformations with generators τ and ζ as just described.

We will give an explicit condition on the generators and q that is equivalent to the invariance condition

$$\lim_{\varepsilon \to 0} \frac{L_\varepsilon \frac{dw_\varepsilon}{dt} - L}{\varepsilon} = 0$$

for the family of transformations applied to this Lagrangian and each time.

Note that for each **fixed time** this is the derivative with respect to ε of the function $L_\varepsilon \frac{dw_\varepsilon}{dt} - L$ evaluated at $\varepsilon = 0$.

THE INVARIANCE IDENTITY

The fact we aim to prove here is:

$$\lim_{\varepsilon \to 0} \frac{L_\varepsilon \frac{dw_\varepsilon}{dt} - L}{\varepsilon} = 0 \qquad \text{(The Invariance Condition)}$$

if and only if

$$\frac{\partial L}{\partial q^i}\zeta^i + p_i \dot{\zeta}^i + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = 0 \qquad \text{(The Invariance Identity)}$$

for every path q, where $\dot{\tau} = \frac{d\tau}{dt}$ and $\dot{\zeta}^i = \frac{d\zeta^i}{dt}$ and p_i is the conjugate momentum $p_i = \frac{\partial L}{\partial \dot{q}^i}$ and H is the Hamiltonian $H = p_i \dot{q}^i - L$.

In this section *none* of the paths are assumed to be stationary.

Proof:

Recall that for each **fixed time** the invariance condition is that the derivative with respect to ε of the function $L_\varepsilon \frac{dw_\varepsilon}{dt} - L$ evaluated at $\varepsilon = 0$ is 0.

L itself is independent of ε and so has derivative 0.

Differentiating the first term by the product rule gives

$$\frac{d}{d\varepsilon}\left(L_\varepsilon \frac{dw_\varepsilon}{dt} - L\right) = \frac{dw_\varepsilon}{dt}\frac{d}{d\varepsilon}L_\varepsilon + L_\varepsilon \frac{d}{d\varepsilon}\frac{dw_\varepsilon}{dt}. \tag{2}$$

Let's begin identifying these four factors evaluated at $\varepsilon = 0$.

$$\frac{dw_\varepsilon}{dt}\frac{d}{d\varepsilon}L_\varepsilon + L_\varepsilon \frac{d}{d\varepsilon}\frac{dw_\varepsilon}{dt} \quad \text{is to be evaluated at } \varepsilon = 0.$$

First, when L_ε is evaluated at $\varepsilon = 0$ we have

$$L_0 = L. \tag{3}$$

Next, as a function of time w_ε is given as $w_\varepsilon = t + \varepsilon\tau + \varepsilon^2 g$ so

$$\frac{dw_\varepsilon}{dt} = 1 + \varepsilon\dot{\tau} + \varepsilon^2\dot{g} \tag{4}$$

which is 1 when evaluated at $\varepsilon = 0$.

Also

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{dw_\varepsilon}{dt} = \dot{\tau}. \tag{5}$$

This gives us 3 out of the 4 factors when evaluated at $\varepsilon = 0$.

All that remains is to calculate the fourth factor $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_\varepsilon$, and we will do that by applying the chain rule.

For each fixed t (remember, ε will be the variable here) we have

$$\begin{aligned} L_\varepsilon(t) &= L\left(w_\varepsilon(t), Q_\varepsilon(w_\varepsilon(t)), \frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon(t))\right) \\ &= L(A, B, C). \end{aligned}$$

$$A = t + \varepsilon\tau(t, q(t)) + \varepsilon^2 f(t, q(t)) \Rightarrow \left.\frac{dA}{d\varepsilon}\right|_{\varepsilon=0} = \tau. \tag{6}$$

$$B = q(t) + \varepsilon\zeta(t, q(t)) + \varepsilon^2 g(t, q(t)) \Rightarrow \left.\frac{dB}{d\varepsilon}\right|_{\varepsilon=0} = \zeta. \tag{7}$$

Going back to Equation (2) we have

$$C = \frac{dQ_\varepsilon}{dw_\varepsilon}(w_\varepsilon(t)) = \frac{\frac{dQ_\varepsilon \circ w_\varepsilon}{dt}}{\frac{dw_\varepsilon}{dt}} = \frac{\frac{d}{dt}q_\varepsilon(t, q(t))}{\frac{dw_\varepsilon}{dt}} = \frac{\dot{q} + \varepsilon\dot{\zeta} + \varepsilon^2\dot{g}}{1 + \varepsilon\dot{\tau} + \varepsilon^2\dot{f}}. \tag{8}$$

We still need to calculate $\left.\frac{dC}{d\varepsilon}\right|_{\varepsilon=0}$ which we do by the quotient rule applied to

$$C = \frac{\dot{q} + \varepsilon\dot{\zeta} + \varepsilon^2\dot{g}}{1 + \varepsilon\dot{\tau} + \varepsilon^2\dot{f}}.$$

Any term not involving ε will differentiate to 0 as that rule is applied. And any term possessing an ε factor *after* the differentiation procedure will evaluate to 0 when ε is finally set to 0.

So we have

$$\left.\frac{dC}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\dot{\zeta}\cdot 1 - \dot{q}\dot{\tau}}{1^2} = \dot{\zeta} - \dot{q}\dot{\tau}. \tag{9}$$

Using Equations (6), (7) and (9) we can calculate the fourth term as

$$\begin{aligned} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_\varepsilon &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(A, B, C) \\ &= \frac{\partial L}{\partial t}\frac{dA}{d\varepsilon} + \frac{\partial L}{\partial q^i}\frac{dB^i}{d\varepsilon} + \frac{\partial L}{\partial \dot{q}^i}\frac{dC^i}{d\varepsilon} \\ &= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\zeta^i + \frac{\partial L}{\partial \dot{q}^i}\left(\dot{\zeta}^i - \dot{q}^i\dot{\tau}\right) \\ &= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\zeta^i + p_i\left(\dot{\zeta}^i - \dot{q}^i\dot{\tau}\right). \end{aligned} \tag{10}$$

Resurrecting the four factors, evaluated at $\varepsilon = 0$, using Equations (3), (4), (5) and (10) we have

$$\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(L_\varepsilon \frac{dw_\varepsilon}{dt} - L\right) \\
&= \frac{dw_\varepsilon}{dt}\frac{d}{d\varepsilon}L_\varepsilon + L_\varepsilon \frac{d}{d\varepsilon}\frac{dw_\varepsilon}{dt} \quad \text{(evaluated at } \varepsilon = 0) \\
&= 1 \cdot \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L_\varepsilon\right) + L_0 \dot{\tau} \\
&= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\zeta^i + p_i(\dot{\zeta}^i - \dot{q}^i\dot{\tau}) + L\dot{\tau} \\
&= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\zeta^i + p_i\dot{\zeta}^i - \left(p_i\dot{q}^i - L\right)\dot{\tau} \\
&= \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\zeta^i + p_i\dot{\zeta}^i - H\dot{\tau}.
\end{aligned}$$

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When

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(L_\varepsilon \frac{dw_\varepsilon}{dt} - L\right) = \frac{dw_\varepsilon}{dt}\frac{d}{d\varepsilon}L_\varepsilon + L_\varepsilon \frac{d}{d\varepsilon}\frac{dw_\varepsilon}{dt} \quad \text{(at } \varepsilon = 0)$$

we have

$$0 = \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\zeta^i + p_i\dot{\zeta}^i - H\dot{\tau}.$$

This is the invariance identity.

These terms can be untangled to produce the 4 terms at the top-right of this slide, so if the invariance identity holds so too does the invariance condition and the theorem is proved. □

Part Two TOC Recap Invariance Noether's 1st Thm The Killing Equations Refs

In many applications the invariance condition is not exactly satisfied but a related condition *is* satisfied that is sufficient to create a conservation law.

Instead of

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(L_\varepsilon \frac{dt_\varepsilon}{dt} - L\right) = 0 \quad \text{for every } q \text{ and every time}$$

we suppose that there is a function F defined on $\mathbb{R} \times \mathbb{R}^N$ for which

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\left(L_\varepsilon \frac{dt_\varepsilon}{dt} - L\right) = \frac{d}{dt}F(t, q(t)) \quad \text{for every } q \text{ and every time.}$$

So a total time derivative $\dot{F} = \frac{dF}{dt}$ is involved in the equation.

We say L is divergence-invariant in this case, or invariant up to the divergence term F.

Part Two TOC Recap Invariance Noether's 1st Thm The Killing Equations Refs

The Divergence-Invariance Identity

This produces a modified equivalence:

$$\lim_{\varepsilon \to 0} \frac{L_\varepsilon \frac{dt_\varepsilon}{dt} - L}{\varepsilon} = \dot{F}$$

if and only if

$$\frac{\partial L}{\partial q^i}\zeta^i + p_i\dot{\zeta}^i + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = \dot{F}$$

for every path q, where $\dot{\tau} = \frac{d\tau}{dt}$ and $\dot{\zeta}^i = \frac{d\zeta^i}{dt}$ and p_i is the conjugate momentum $p_i = \frac{\partial L}{\partial \dot{q}^i}$ and $H = p_i\dot{q}^i - L$.

This one is called the divergence-invariance identity. The original invariance identity is a special case where $F = 0$.

In this single-independent-variable case[4] you can, obviously, concoct an antiderivative for the left-hand-side of

$$\dot{p}_i\zeta^i + p_i\dot{\zeta}^i - \dot{H}\tau - H\dot{\tau} = \dot{F}$$

to create F for each particular solution q.

This is not interesting—*unless* F can be recognized as a function of the coordinates $(t, q(t))$, independent of the particular solution q you select.

[4] In the higher-dimensional case we will be looking for a function whose divergence is specified by a similar formula, hence the name "divergence-invariance" for the condition.

As a side note there is an interesting relationship between Φ_ε and F. You may recall that before we began considering

$$\frac{d}{d\varepsilon}\left(L_\varepsilon \frac{dw_\varepsilon}{dt} - L\right)$$

at individual times along a path q we were concerned with the global condition

$$\Phi_\varepsilon(q) - \Phi_0(q) = \int_a^b \left(L_\varepsilon \frac{dw_\varepsilon}{dt} - L\right) dt$$

on the whole path q.

If L is invariant up to divergence term F then

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi_\varepsilon(q) = \int_a^b \frac{d}{dt}F(t, q(t))\, dt = F(b, q(b)) - F(a, q(a)).$$

So the derivative of $\Phi_\varepsilon(q)$ at $\varepsilon = 0$, though defined as an integral using the entire curve q, depends only on the endpoints of this path.

We are now ready to prove Noether's Theorem for one independent variable. This is merely a matter of rearranging the divergence-invariance identity and making an observation.

We assume

$$\frac{\partial L}{\partial q^i}\zeta^i + p_i\dot{\zeta}^i + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = \dot{F}$$

and that q is a stationary path so the Euler-Lagrange identities hold:

$$\frac{\partial L}{\partial t} = -\dot{H} \quad \text{and} \quad \frac{\partial L}{\partial q^i} = \dot{p}_i.$$

This turns the divergence-invariance identity into:

$$\dot{p}_i\zeta^i + p_i\dot{\zeta}^i - \dot{H}\tau - H\dot{\tau} = \dot{F}.$$

NOETHER'S FIRST THEOREM (ONE INDEPENDENT VARIABLE)

If L is invariant up to the divergence term F for the family of transformations with generators τ and ζ, and if q is a stationary path then

$$H\tau + F - p_i\zeta^i \quad \text{is constant on path } q.$$

Given:

$$\dot{p}_i\zeta^i + p_i\dot{\zeta}^i - \dot{H}\tau - H\dot{\tau} = \dot{F}.$$

we have:

$$0 = -\dot{p}_i\zeta^i - p_i\dot{\zeta}^i + \dot{H}\tau + H\dot{\tau} + \dot{F} \Longrightarrow 0 = \frac{d}{dt}\left(H\tau + F - p_i\zeta^i\right).$$

The theorem follows. □

Example: The Arclength Functional

Let's see what all this says in a very simple example with the arclength Lagrangian $L(t, x) = \sqrt{1 + \dot{x}^2}$.

$p = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}$.

$H = p\dot{x} - \sqrt{1 + \dot{x}^2} = \frac{\dot{x}^2-(1+\dot{x}^2)}{\sqrt{1+\dot{x}^2}} = \frac{-1}{\sqrt{1+\dot{x}^2}}$.

$\frac{\partial L}{\partial t} = 0$ and $\frac{\partial L}{\partial x} = 0$ so both H and p will be preserved along stationary solutions. Either condition implies that stationary solutions for this Lagrangian are straight lines.

Let's see what Noether's theorem says about this when the family of transformations is given by

$$t_\varepsilon = t - \varepsilon x \Rightarrow \tau = -x$$
$$x_\varepsilon = x + \varepsilon t \Rightarrow \zeta = t.$$

$$t_\varepsilon = t - \varepsilon x \Rightarrow \tau = -x$$
$$x_\varepsilon = x + \varepsilon t \Rightarrow \zeta = t.$$

We need to evaluate

$$\frac{\partial L}{\partial x}\zeta + p\dot{\zeta} + \frac{\partial L}{\partial t}\tau - H\dot{\tau}.$$

This gives

$$0 \cdot t + \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} \cdot 1 + 0 \cdot (-x) - (\frac{-1}{\sqrt{1+\dot{x}^2}})(-\dot{x}) = 0.$$

So the Lagrangian is invariant under this family of transformations.

So the following expression is constant on solution paths.

$$H\tau - p\zeta = \frac{-1}{\sqrt{1+\dot{x}^2}}(-x) - \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}t = \frac{x - t\dot{x}}{\sqrt{1+\dot{x}^2}}.$$

This might be surprising . . . *if* we didn't already know that stationary paths for this Lagrangian are linear.

I guess this example is a little too easy to impress anyone, so let's try another.

Projectile Motion in a Verticle Plane

A sensible Lagrangian here is $L(y, \dot{y}, \dot{x}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$.

$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$.

$$H = p_x\dot{x} + p_y\dot{y} - L = m\dot{x}^2 + m\dot{y}^2 - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy$$
$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy.$$

$\frac{\partial L}{\partial t} = 0$ and $\frac{\partial L}{\partial x} = 0$ so both H and p_x will be preserved along stationary solutions. But $\frac{\partial L}{\partial y} = -mg$ so p_y is not constant on solutions.

Consider the family of transformations is given by

$$t_\varepsilon = t \Rightarrow \tau = 0$$
$$x_\varepsilon = x - \varepsilon t \Rightarrow \zeta_x = -t$$
$$y_\varepsilon = y \Rightarrow \zeta_y = 0.$$

We need to evaluate

$$\frac{\partial L}{\partial x}\zeta_x + \frac{\partial L}{\partial y}\zeta_y + p_x\dot{\zeta}_x + p_y\dot{\zeta}_y + \frac{\partial L}{\partial t}\tau - H\dot{\tau}.$$

This gives

$$0\cdot(-t) + (-mg)\cdot 0 + m\dot{x}(-1) + m\dot{y}\cdot 0 + 0\cdot 0 - H\cdot 0 = -m\dot{x}.$$

So the Lagrangian is divergence-invariant under this family of transformations with total derivative $-m\dot{x}$, so the divergence term is $F(x) = -mx$.

In this situation, the conserved quantity is

$$\begin{aligned} &H\tau + F - p_x\zeta_x - p_y\zeta_y = H\cdot 0 - mx - p_x(-t) - p_y\cdot 0 \\ &= -mx + m\dot{x}t. \end{aligned}$$

Example: A Damped Oscillator

Here we have Lagrangian $L(t,x,\dot{x}) = \left(\dot{x}^2 - x^2\right)e^t$.

$p = \frac{\partial L}{\partial \dot{x}} = 2\dot{x}e^t$.

$H = p\dot{x} - L = 2\dot{x}e^t\dot{x} - \left(\dot{x}^2 - x^2\right)e^t = \left(\dot{x}^2 + x^2\right)e^t$.

$\frac{\partial L}{\partial t} = L$ and $\frac{\partial L}{\partial x} = -2xe^t$ so probably neither H nor p will be preserved along stationary solutions.

Let's see what Noether's theorem says about this one:

$$\begin{gathered} t_\varepsilon = t + \varepsilon \Rightarrow \tau = 1 \\ x_\varepsilon = x - \frac{\varepsilon}{2}x \Rightarrow \zeta = -\frac{x}{2}. \end{gathered}$$

We need to evaluate

$$\frac{\partial L}{\partial x}\zeta + p\dot{\zeta} + \frac{\partial L}{\partial t}\tau - H\dot{\tau}.$$

$$-2xe^t\left(-\frac{x}{2}\right) + 2\dot{x}e^t\left(-\frac{\dot{x}}{2}\right) + \left(\dot{x}^2 - x^2\right)e^t\cdot 1 - \left(\dot{x}^2 + x^2\right)e^t\cdot 0 = 0.$$

This Lagrangian is invariant under this family of transformations.

So the following expression is constant on solution paths.

$$H\tau - p\zeta = \left(\dot{x}^2 + x^2\right)e^t\cdot 1 - 2\dot{x}e^t\left(-\frac{x}{2}\right) = \left(\dot{x}^2 + x^2 + \dot{x}x\right)e^t.$$

So exactly what does this conserved quantity represent? This is a very interesting physics question. Another interesting question, this one mathematical, is how someone might *find* the generators that produced this invariant quantity.

The Killing Equations[5]

Known constants-of-an-interaction such as mass-energy or spin or electric charge or strangeness or charm put restraints on the type of Lagrangian (or Lagrangian density) that governs the physics, and so are useful if you are creating a new theory (i.e. creating a governing Lagrangian) such as the Standard Model of particle physics.

I was told (possibly it's even true) that this particular theory was bolted together ad hoc by Murray Gell-Mann and his associates using just such considerations, and is tested by experiments using these ideas.

On the other hand, if you are given a known Lagrangian it is not always clear how to find generators of symmetries for it.

We will consider this aspect now.

[5] Named after Wilhem Killing (1847-1923).

The underlying idea is fairly simple, and will produce *necessary* conditions on generators that quite often are sufficient for invariance of the Lagrangian with respect to transformations you actually produce, and to rule out the possibility of invariant families of certain types.

The invariance identity

$$\frac{\partial L}{\partial q^i}\zeta^i + p_i\dot{\zeta}^i + \frac{\partial L}{\partial t}\tau - H\dot{\tau} = \dot{F}$$

must hold for *any* path q including, for instance, those paths for which $\dot{q}^i = 1$ for a particular i and $q^k = 0$ for $k \neq i$.

Replacing H by $p_i\dot{q}^i - L$ and p_i by $\frac{\partial L}{\partial \dot{q}^i}$ we have

$$\frac{\partial L}{\partial q^i}\zeta^i + \frac{\partial L}{\partial \dot{q}^i}\dot{\zeta}^i + \frac{\partial L}{\partial t}\tau - \left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right)\dot{\tau} = \dot{F}.$$

$$\frac{\partial L}{\partial q^i}\zeta^i + \frac{\partial L}{\partial \dot{q}^i}\dot{\zeta}^i + \frac{\partial L}{\partial t}\tau - \left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right)\dot{\tau} = \dot{F}.$$

Recall $\tau = \tau(t, q)$ and $\zeta = \zeta(t, q)$ so

$$\dot{\tau} = \frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q^k}\dot{q}^k \quad \text{and} \quad \dot{\zeta}^i = \frac{\partial \zeta^i}{\partial t} + \frac{\partial \zeta^i}{\partial q^k}\dot{q}^k.$$

Substituting these into the invariance equation produces

$$\begin{aligned}\dot{F} = \frac{\partial L}{\partial q^i}\zeta^i &+ \frac{\partial L}{\partial \dot{q}^i}\left(\frac{\partial \zeta^i}{\partial t} + \frac{\partial \zeta^i}{\partial q^k}\dot{q}^k\right)\\ &+ \frac{\partial L}{\partial t}\tau - \left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right)\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q^k}\dot{q}^k\right).\end{aligned}$$

Expanding and re-ordering the eight terms gives

$$\begin{aligned}\dot{F} = L\frac{\partial \tau}{\partial t} + L\frac{\partial \tau}{\partial q^k}\dot{q}^k + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\zeta^i + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \zeta^i}{\partial t}\\ + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \zeta^i}{\partial q^k}\dot{q}^k - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\frac{\partial \tau}{\partial t} - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\frac{\partial \tau}{\partial q^k}\dot{q}^k.\end{aligned}$$

This provides numerous necessary relationships that must be satisfied by τ and the ζ^i, revealed by particular choices of q in conjunction with guesses about generators τ and ζ, such as $\tau = 0$ or $\tau = 1$.

These necessary relationships can suggest a path to generators of families of transformations for which L is invariant.

Let's examine an example from classical physics, a Newtonian particle subject to Lagrangian with time-dependent potential

$$L(t, x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - U(t, x).$$

So $\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$ and $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $\frac{\partial L}{\partial t} = -\frac{\partial U}{\partial t}$.

Feeding this into the invariance identity with $\dot{F} = 0$ yields

$$\begin{aligned}0 &= L\frac{\partial \tau}{\partial t} + L\frac{\partial \tau}{\partial q^k}\dot{q}^k + \frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^i}\zeta^i + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \zeta^i}{\partial t} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \zeta^i}{\partial q^k}\dot{q}^k - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\frac{\partial \tau}{\partial t} - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\frac{\partial \tau}{\partial q^k}\dot{q}^k\\ &= \left(\frac{1}{2}m\dot{x}^2 - U\right)\frac{\partial \tau}{\partial t} + \left(\frac{1}{2}m\dot{x}^2 - U\right)\frac{\partial \tau}{\partial x}\dot{x} - \frac{\partial U}{\partial t}\tau - \frac{\partial U}{\partial x}\zeta + m\dot{x}\frac{\partial \zeta}{\partial t} + m\dot{x}\frac{\partial \zeta}{\partial x}\dot{x} - m\dot{x}\dot{x}\frac{\partial \tau}{\partial t} - m\dot{x}\dot{x}\frac{\partial \tau}{\partial x}\dot{x}\\ &= \left(-U\frac{\partial \tau}{\partial t} - \frac{\partial U}{\partial t}\tau - \frac{\partial U}{\partial x}\zeta\right) + \left(-U\frac{\partial \tau}{\partial x} + m\frac{\partial \zeta}{\partial t}\right)\dot{x} + \left(m\frac{\partial \zeta}{\partial x} - \frac{1}{2}m\frac{\partial \tau}{\partial t}\right)\dot{x}^2 + \left(-\frac{1}{2}m\frac{\partial \tau}{\partial x}\right)\dot{x}^3.\end{aligned}$$

Since path x is arbitrary, each coefficient of this polynomial in $\dot{x}$ must be the zero function. Setting these four coefficients to 0, individually, yields the four Killing equations for this Lagrangian.

$$0 = \left(-U\frac{\partial\tau}{\partial t} - \frac{\partial U}{\partial t}\tau - \frac{\partial U}{\partial x}\zeta\right) + \left(-U\frac{\partial\tau}{\partial x} + m\frac{\partial\zeta}{\partial t}\right)\dot{x} + \left(m\frac{\partial\zeta}{\partial x} - \frac{1}{2}m\frac{\partial\tau}{\partial t}\right)\dot{x}^2 + \left(-\frac{1}{2}m\frac{\partial\tau}{\partial x}\right)\dot{x}^3.$$

The third degree term tells us τ must be a function of t alone. The first degree term requires, then, that ζ depends on x alone.

The second degree term tells us, then, that both $\frac{d\zeta}{dx}$ and $\frac{d\tau}{dt}$ must be constant with $\frac{d\tau}{dt}$ twice as big as $\frac{d\zeta}{dx}$.

Let's suppose $\frac{d\zeta}{dx} = C$. Then $\zeta = Cx + B$ and $\frac{d\tau}{dt} = 2C$ and $\tau = 2Ct + A$, for constants A and B.

The first Killing equation

$$-U\frac{\partial\tau}{\partial t} - \frac{\partial U}{\partial t}\tau - \frac{\partial U}{\partial x}\zeta = 0$$

$$-U2C - \frac{\partial U}{\partial t}(2Ct + A) - \frac{\partial U}{\partial x}(Cx + B) = 0$$

puts severe restriction on the form of U—remember, we *have* U in this scenario. If $C = 0$ we must have

$$\frac{\partial U}{\partial t}A = -\frac{\partial U}{\partial x}B$$

for constants A and B. And if $C \neq 0$ we must have

$$U = -\frac{\partial U}{\partial t}\left(\frac{2Ct + A}{2C}\right) - \frac{\partial U}{\partial x}\left(\frac{Cx + B}{2C}\right)$$

for constants A and B.

These conditions can either be satisfied for certain constant A, B and C or not. If not there are no invariant families for this Lagrangian. If yes we have generators for an invariant family.

The modification to allow for divergence-invariance requires modifying the first Killing equation to

$$-U\frac{\partial\tau}{\partial t} - \frac{\partial U}{\partial t}\tau - \frac{\partial U}{\partial x}\zeta = \dot{F}$$

$$-U2C - \frac{\partial U}{\partial t}(2Ct + A) - \frac{\partial U}{\partial x}(Cx + B) = \dot{F}$$

for some differentiable function F.

We may be able to identify the left-hand-side as a total derivative on arbitrary path x of some function $F = F(t, x(t))$ for certain constant A, B and C.

If we can do that, we then proceed to find invariants via Noether's Theorem in the divergence-invariance setting too.

References

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